

# Five-Brane Thermodynamics from the Matrix Model

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**ABSTRACT:** A certain sector of the matrix model for M-theory on a plane wave background has recently been shown to produce the transverse five-brane. We consider this theory at finite temperature. We find that, at a critical temperature it has a Gross-Witten phase transition which corresponds to deconfinement of the matrix model gauge theory. We interpret the phase transition as the Hagedorn transition of M-theory and of type II string theory in the five-brane background. We also show that there is no Hagedorn behaviour in the transverse membrane background case.

**KEYWORDS:** Penrose limit and pp-wave background, String Duality.

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## 1. Introduction and Summary

The matrix theory conjecture states that the large  $N$  limit of a certain matrix quantum mechanics is an exact description of light-cone quantized M-theory in flat eleven-dimensional space-time [1] (for a review see [2]). Many of the objects of M-theory, such as the supergraviton, supermembrane and M five-brane emerge in a natural way from the matrix model and some of their low-energy interactions have been computed and matched with

supergravity. However, the study of this seemingly simple model has been plagued by its nonlinearity and the existence of flat directions in the potential leading to a continuous spectrum [3].

Recently a matrix model of M-theory in an 11-dimensional plane-wave geometry has been studied [4]-[10]. It has the great advantage that one can formulate an accurate perturbative expansion of the model. Properties of states and symmetries have been analyzed in this limit [5]-[7]. It has also been realized that some of the states are protected by supersymmetry.

The plane-wave background has a single dimensional parameter  $\mu$  which appears in the metric and the constant four-form field strength,

$$ds^2 = -2dx^+dx^- - \left( \left( \frac{\mu}{3} \right)^2 (x^a)^2 + \left( \frac{\mu}{6} \right)^2 (x^{a'})^2 \right) (dx^+)^2 + (dx^a)^2 + (dx^{a'})^2$$

$$F_{+123} = \mu \quad (1.1)$$

This is a 1-parameter family of backgrounds which goes to Minkowski space when  $\mu \rightarrow 0$ . The matrix model for M-theory on this background has action [4]

$$S = R \int dt \text{Tr} \left\{ \frac{1}{2R^2} (D_t X^i)^2 + \frac{i}{R} \psi^t D_t \psi + \psi^t \Gamma^i [X^i, \psi] + \frac{1}{4} [X^i, X^j]^2 \right.$$

$$\left. - \frac{1}{2} \left( \frac{\mu}{3R} \right)^2 (X^a)^2 - \frac{1}{2} \left( \frac{\mu}{6R} \right)^2 (X^{a'})^2 - i \frac{\mu}{4R} \psi^t \Gamma^{123} \psi - i \frac{\mu}{3R} \epsilon_{abc} X^a X^b X^c \right\} \quad (1.2)$$

We have set the M-theory Planck length to one,  $\ell_p = 1$ , and the conventions for indices are  $i, j, \dots = 1, \dots, 9$  and  $a, b, \dots = 1, 2, 3$ ,  $a', b', \dots = 4, \dots, 9$ . All degrees of freedom are  $N \times N$  Hermitian matrices. The covariant derivative is

$$D_t X = \frac{d}{dt} X + i[A, X] \quad (1.3)$$

This matrix model is conjectured to be a discrete light-cone quantization (DLCQ) of M-theory where  $R$  is the compactification radius of the null coordinate  $x^-$  and there are  $N$  units of light-cone momentum,  $p^+ = N/R$ . It is a 1-parameter deformation of the BFSS matrix model [1] which described M-theory on Minkowski space. Just as the BFSS model could also be regarded as the effective action describing the low energy dynamics of a collection of  $N$  D0-branes of type IIA string theory on Minkowski space, with the appropriate re-interpretation of the parameters, (1.2) could also describe D0-branes in a background of type IIA string theory.

The matrix model can be analyzed in perturbation theory. As was noted in ref. [4], one interesting feature of the semiclassical expansion is the existence of a large number of different vacua. The potential contains a term of the form

$$V \sim \text{Tr} \left( \frac{\mu}{3R} X^a + i \epsilon^{abc} X^b X^c \right)^2 \quad (1.4)$$

Classical vacua are

$$X^a(t) = \frac{\mu}{3R} J^a \quad (1.5)$$

where  $J^a$  are matrices which satisfy the  $SU(2)$  Lie algebra,

$$[J^a, J^b] = i\epsilon^{abc} J^c \quad (1.6)$$

and could be either irreducible or reducible representations of  $SU(2)$ . The representations of  $SU(2)$  are interpreted as fuzzy spheres which are spherically symmetric states of membranes [4] (see also [8, 9]). They approach a classical membrane when the spin of an irreducible representation is large, the state with a single membrane being described by a single  $N$ -dimensional irreducible representation of  $SU(2)$  with spin  $j = \frac{N-1}{2}$ .

Recently, an alternative interpretation of these solutions as transverse five-branes has been given [10]. In principle, any of the classical vacua corresponds to a five-brane state. A single classical five-brane, with its five spatial dimensions forming a five-sphere and its time direction lying along the light-cone, is believed to correspond to the vacuum where the representation of  $SU(2)$  is highly reducible, consisting of  $N$  repetitions of the singlet  $J^i = 0$  representation. A detailed comparison of the fluctuation spectrum of the matrix model about these states with the fluctuations of the classical five-brane on a plane-wave background was given in ref. [10] and good agreement was found. A stack of  $k$  coincident five-branes is thought to correspond to the state with  $n = N/k$  repetitions of the  $k$ -dimensional representation of  $SU(2)$ . If the representations have differing dimensionalities, with maximum dimension  $k$ , this still corresponds to  $k$  spherical five-brane states where the branes have different radii.

The main difference between a five-brane state and a membrane state is that, for the five-brane, as  $N$  is taken to infinity, the number of repetitions of irreducible representations of  $SU(2)$  is taken to infinity with their dimensionalities held fixed, whereas for the membrane, the number or repetitions of an irreducible representation (number of membranes) is held fixed and the dimensionalities are taken to infinity.

The distinction between these two cases that will be important to us in the following occurs in the way that the gauge symmetry of (1.2) is realized. Vacua with non-trivial representations of  $SU(2)$  break the gauge symmetry, realizing it in a Higgs phase. There is a residual gauge symmetry which interchanges representations of the same dimensionality. If there are  $n_k$  representations of dimension  $k$ , the gauge symmetry breaking pattern is

$$\frac{U(N)}{U(1)} \rightarrow \prod_{k=1}^{\infty} \left( \frac{U(n_k)}{U(1)} \right) \quad (1.7)$$

Here, because all degrees of freedom transform in the adjoint representation, the true gauge group is the group modulo its center. For a membrane state, as we take the limit  $N \rightarrow \infty$ , the rank of the residual gauge group remains finite, whereas for a five-brane, it becomes infinite.

The coupling constant which governs the size of corrections to perturbation theory depends on the classical vacuum about which one is expanding. For membrane vacua, the effective coupling constant is

$$g_{\text{eff}} = \left( \frac{3R}{\mu} \right)^{\frac{3}{2}} \quad (1.8)$$

For the five-brane, where the multiplicity of representations is  $n$ , the effective coupling is the 't Hooft coupling

$$g_{\text{eff}}^2 n = \lambda \equiv \left( \frac{3R}{\mu} \right)^3 n \quad (1.9)$$

The difference comes from the residual  $U(n)$  symmetry of the five-brane solution and the fact that index loops in perturbation theory contribute factors of  $n$  which can only be controlled in the 't Hooft limit,

$$n \rightarrow \infty \quad \text{with} \quad \lambda \text{ fixed} \quad (1.10)$$

The decoupling limit, which isolates the theory living on the five-brane, is the same as the 't Hooft limit in (1.10). Intuitively, this is the weakest coupling limit that can be taken that retains any of the interactions in the matrix model. If it were taken for a membrane state, the matrix model would be non-interacting. It is only the five-brane states which retain interactions. The 't Hooft coupling is the radius of the five-brane squared in units of the string scale

$$\frac{r^2}{\alpha'} \sim \lambda \quad (1.11)$$

In this work we will be concerned with perturbative computations in five-brane backgrounds in the 't Hooft limit. Our perturbative analysis is limited to small  $\lambda$ , the situation where the five-brane is highly curved. In the end this will prevent us from making a direct comparison of our results with the properties of the usual NS five-brane of string theory which is large and flat and which we believe would be obtained by the infinite radius, and therefore infinite 't Hooft coupling limit, in these sectors of the matrix model.

In this paper we shall analyze the thermodynamics of this matrix model. The main observation will be that the five-brane states of the matrix model have a first order phase transition at some critical temperature. This phase transition is seen in the 't Hooft limit. Our analysis is further limited to the regime where the 't Hooft coupling  $\lambda$  is small. However, the existence of the phase transition appears robust and is expected to occur over a range of  $\lambda$ .

We will show this in two different ways. The first way counts the degeneracy of states with high energies in the infinite  $N$  limit and establishes that there is a Hagedorn density of states. From the exponential behavior we can find the Hagedorn temperature.

Then, secondly, we use matrix model techniques to show that there is a Gross-Witten type of large  $N$  phase transition in the perturbative limit of the model. In the weak coupling limit, the transition temperature coincides with the Hagedorn temperature that we extracted from the asymptotic density of states. The advantage of the matrix model approach is that it is at least in principle possible to study the phase above the Hagedorn transition. The transition is similar to deconfinement transitions which are expected to occur in confining gauge theories at high temperature.

### Note Added

While this research was conducted, an extensive study of the Hagedorn transition in the

case of weakly coupled  $SU(N)$  gauge theories on compact spatial manifolds was carried out in an interesting work [36]. This work has a partial technical overlap with ours.

## 2. Hagedorn Behaviour of Matrix Models

In this section, we will compute the Hagedorn temperature of matrix models by counting their gauge invariant states, or, equivalently, the numbers of gauge invariant operators. We will show that, when the dimension  $N$  of the matrices is infinite, they have a Hagedorn density of states. Our analysis is very similar to that in ref. [11, 12, 13].

### 2.1 A Toy Model

We will begin with a toy model which captures the salient features of the physics involved. Consider a set of  $d$  bosonic  $N \times N$  Hermitian matrices,  $X_j(t)$ , where  $j = 1, \dots, d$  and  $t$  is the time. These are coupled to an  $N \times N$  “gauge field”  $A(t)$ . The action is

$$S = \int dt \sum_{j=1}^d \frac{1}{2} \left( \left( \frac{d}{dt} X_j + i[A, X_j] \right)^2 - \omega^2 X_j^2 \right) \quad (2.1)$$

If it were not for the gauge field appearing in the covariant derivative this would be a theory of  $d$  independent matrix oscillators. Introducing the gauge field makes the model invariant under the gauge transformation

$$\begin{aligned} X_j(t) &\rightarrow U(t) X_j(t) U^\dagger(t) \\ A(t) &\rightarrow U(t) A(t) U^\dagger(t) - i U(t) \dot{U}^\dagger(t) \end{aligned} \quad (2.2)$$

The equation of motion which is obtained by varying  $A(t)$  has no time derivatives and is an equation of constraint

$$\mathcal{G} = \sum_{j=1}^d [X_j(t), \Pi_j(t)] \sim 0 \quad (2.3)$$

where

$$\Pi_j(t) = \frac{d}{dt} X_j(t) + i[A(t), X_j(t)] \quad (2.4)$$

is the canonical momentum, with Poisson bracket

$$\left\{ X_j^{ab}(t), \Pi_k^{cd}(t) \right\} = \delta_{jk} \delta^{ad} \delta^{bc} \quad (2.5)$$

The left-hand-side of the constraint equation (2.3) is the generator of an infinitesimal time-independent gauge transformation,

$$\delta X_j = \{ \text{Tr}(\Lambda \mathcal{G}), X_j \} = -[\Lambda, X_j] \quad (2.6)$$

When we quantize, this toy model is simply described by  $d$  decoupled matrix oscillators with the constraint (2.3) on the physical states.<sup>1</sup>

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<sup>1</sup>Obviously, in a deconfined phase this no longer holds for the vacuum as the center symmetry is broken.

**Claim.** *In the limit  $N \rightarrow \infty$  this toy model has a Hagedorn density of states and Hagedorn temperature*

$$T_H = \omega / \ln d \quad (2.7)$$

*The case  $d = 1$  should be interpreted as  $T_H = \infty$ . We will also use the parameter  $\beta = 1/T$  for the inverse temperature and the notation  $\beta_H = 1/T_H = \ln d / \omega$ .*

To substantiate this claim, we will count the number of physical states which have a given asymptotically large energy and extract  $\beta_H$  from the exponential growth (if there is any).

The Hamiltonian which follows from canonical quantization of the action (2.1) is

$$H = \sum_{j=1}^d \frac{1}{2} \text{Tr} (\Pi_j^2 + \omega^2 X_j^2) \quad (2.8)$$

where the momentum obeys the canonical commutation relation

$$[X_j^{ab}, \Pi_k^{cd}] = i \delta_{jk} \delta^{ad} \delta^{bc} \quad (2.9)$$

It is easy to see that the constraint (2.3) commutes with the Hamiltonian.

We shall take the strategy of quantizing the model of free oscillators with Hamiltonian (2.8) and commutators (2.9). From the resulting quantum states we then isolate a subspace of physical states which are annihilated by the constraint operator in equation (2.3).

We define creation and annihilation operators

$$a_j = \frac{1}{\sqrt{2\omega}} (\Pi_j - i\omega X_j) \quad , \quad a_j^\dagger = \frac{1}{\sqrt{2\omega}} (\Pi_j + i\omega X_j) \quad (2.10)$$

so that

$$[a_j^{ab}, a_k^{\dagger cd}] = \delta_{jk} \delta^{ad} \delta^{bc} \quad (2.11)$$

and, after dropping the ground state energy, the Hamiltonian is

$$H = \sum_{j=1}^d \omega \text{Tr} (a_j^\dagger a_j) \quad (2.12)$$

The vacuum state is annihilated by all of the annihilation operators

$$a_j^{ab} |0\rangle = 0 \quad \forall a, b = 1, \dots, N; j = 1, \dots, d \quad (2.13)$$

A basis of quantum states is found by operating creation operators on the vacuum,

$$|0\rangle \quad , \quad a_j^{\dagger ab} |0\rangle \quad , \quad a_j^{\dagger ab} a_k^{\dagger cd} |0\rangle \dots \quad (2.14)$$

These basis vectors are eigenstates of the Hamiltonian.

From these states, we must choose a subspace which is annihilated by the constraint operator in (2.3). That operator generates infinitesimal time independent gauge transformations. Quantum states will be invariant if they are invariant under the finite transform

$$a_j \rightarrow U a_j U^\dagger \quad , \quad a_j^\dagger \rightarrow U a_j^\dagger U^\dagger \quad , \quad \text{with } U U^\dagger = 1 \quad (2.15)$$

Such invariant states are generally obtained from the general basis states by taking traces over the matrix indices,<sup>2</sup>

$$|0\rangle \quad , \quad \text{Tr} \left( a_j^\dagger \right) |0\rangle \quad , \quad \text{Tr} \left( a_j^\dagger a_k^\dagger \right) |0\rangle \quad , \quad \text{Tr} \left( a_j^\dagger \right) \text{Tr} \left( a_k^\dagger \right) |0\rangle \dots \quad (2.16)$$

The operators which create gauge invariant states are thus of the form

$$\text{Tr} \left( a_{j_1}^\dagger \dots a_{j_n}^\dagger \right)$$

and can be thought of as “words” which are made from the “letters”  $a_i^\dagger$ . The length of a word is the number of letters which it contains. There are  $d$  different letters at our disposal. Inside a word, the ordering of the letters is important, since they are matrices and do not commute with each other. However, because of the cyclicity of the trace, words which are related by cyclic permutations of the letters are equivalent.

Note that for finite  $N$  the words are not independent. Generally, a very long word can be expressed in terms of linear combinations of sentences made from shorter words: in the  $U(N)$  theory, the operators  $\text{Tr}(a^\dagger)^k$  for  $k > N$  can be expressed in terms of those with  $0 \leq k \leq N$ . However, the long words do become independent when the limit  $N \rightarrow \infty$  is taken. Here, we shall always assume that the large  $N$  limit is to be taken. For this reason, we shall treat all distinct words as independent operators.

Finally, we can create a state by applying several “words” to the vacuum. Such multi-trace operators,

$$\text{Tr} \left( a_{j_1}^\dagger \dots a_{j_n}^\dagger \right) \text{Tr} \left( a_{k_1}^\dagger \dots a_{k_m}^\dagger \right) \dots \text{Tr} \left( a_{\ell_1}^\dagger \dots a_{\ell_p}^\dagger \right)$$

will be called “sentences”. The length of a sentence is the sum of the lengths of its constituent words, or the total number of letters in the sentence. As each creation operator, or letter, creates an excitation of energy  $\omega$ , the energy is the length of the sentence in units of  $\omega$ .

Obviously, words or sentences should be considered equal if they produce the same quantum state. As operators, words commute with each other. Therefore, the order of the words in a sentence does not make any difference.

In order to find the Hagedorn temperature we should find the density of states at asymptotically large energy. If the theory has a Hagedorn spectrum, the increase in the density of states will be exponential,

$$\rho(E) \sim e^{\beta_H E} \quad (2.17)$$

This formula can be corrected by prefactors; we shall discuss them later, at subsection 2.2.4. We will concentrate for the time being on extracting the coefficient in the exponent which we shall identify with the inverse of the Hagedorn temperature.

To find the density of states, we are essentially looking for the number  $s_n$  of sentences of length  $n$ . In order to compute this number, we will need to consider also the number  $w_k$  of words of length  $k$ . It is convenient to define

$$s_0 = w_0 = 1 \quad (2.18)$$

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<sup>2</sup>If  $N$  were finite there would also be the possibility of other invariants such as the determinant of  $a_j^\dagger$ . The energy of the state created by this operator is  $N\omega$ .



As is common in combinatorial problems, it is helpful to define the generating functions of those numbers,

$$W(x) \equiv \sum_{k=0}^{\infty} w_k x^k \quad (2.19)$$

$$S(x) \equiv \sum_{n=0}^{\infty} s_n x^n \quad (2.20)$$

### 2.1.1 The $d = 1$ Case

For the case  $d = 1$  of an alphabet with a single letter  $a^\dagger \equiv a_1^\dagger$ , there is obviously only one word  $\text{Tr}(a^\dagger)^k$  of each length  $k$ , or  $w_k = 1$ . A sentence of length  $n$  is nothing but a collection of such words, where their order is immaterial. Hence, the number of sentences of length  $n$  is just the number of partitions of  $n$  into natural numbers,  $s_n = p(n)$ . It is well known from the days of Euler that

$$\begin{aligned} P(x) &\equiv \sum_{n=0}^{\infty} p(n) x^n = (1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots) \dots \\ &= \prod_{i=1}^{\infty} (1 - x^i)^{-1} \end{aligned} \quad (2.21)$$

and from the celebrated work of Hardy and Ramanujan that

$$p(n) \sim e^{c\sqrt{n}} \quad (2.22)$$

with  $c = \sqrt{\frac{2}{3}}\pi \approx 2.5651$ .<sup>3</sup>

The number of sentences of length  $n$  is therefore sub-exponential, and the partition function converges for all finite temperatures; consequently we conclude that  $\beta_H = 0$  and in this case  $T_H = \infty$  as claimed.

### 2.1.2 The $d > 1$ Case

An upper bound on the number of words of length  $k$  is

$$w_k \leq d^k$$

as there are  $d$  independent choices for each letter. The cyclic property of the trace makes some of those words equivalent. Obviously, each word can have at most  $k$  equivalent representations; therefore, a lower bound on the number of inequivalent words is

$$w_k \geq \frac{1}{k} d^k$$

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<sup>3</sup>In fact, the result is much stronger. In particular [14],

$$p(n) = \frac{1}{4\sqrt{3}\tilde{n}} \left(1 - \frac{1}{c\sqrt{\tilde{n}}}\right) e^{c\sqrt{\tilde{n}}} + O\left(\frac{1}{\tilde{n}} e^{c\sqrt{\tilde{n}}/2}\right),$$

where

$$\tilde{n} = n - \frac{1}{24}.$$

We claim that the latter bound is in fact a good approximation,

$$w_k \approx \frac{1}{k} d^k$$

To substantiate this claim, we will utilize the famous Polya theorem.

**Theorem (Polya).** *Let  $G$  be a group acting on a set  $X$ . Then the number of colourings of the elements of  $X$  with  $d$  colours, distinct under the action of  $G$ , is*

$$\frac{1}{|G|} \sum_{g \in G} d^{\lambda(g, X)} \quad (2.23)$$

where  $\lambda(g, X)$  is the number of cycles in the action of  $g$  on  $X$ .

In our case,  $X = \mathbb{Z}_k$  is the string of letters in a word, and the group of rotations is also  $G = \mathbb{Z}_k$ , acting on  $X$  by addition. The colours are the different kinds of letters, so their number is indeed  $d$ . The number of colourings is just the number of inequivalent words.

Let us begin with the case when  $k = p$  is a prime number. Then, the identity has  $|X| = p$  orbits, and each other rotation has only one; therefore,<sup>4</sup>

$$w_p = \frac{1}{p} (d^p + (p-1)d) \quad (2.24)$$

This is indeed very close, for large  $p$ , to  $\frac{1}{p} d^p$ . For the case where  $k$  is not a prime, a similar formula can be written in terms of Euler's totient function, but it suffices to note that apart from the identity, again having  $|X| = k$  orbits, all other rotations can have at most  $k/2$  orbits (in fact, there can be at most one such orbit, for  $g = k/2$  when  $k$  is even). Therefore

$$w_k \leq \frac{1}{k} (d^k + (k-1)d^{k/2})$$

which is again close to  $\frac{1}{k} d^k$ .

Accordingly, we shall henceforth use the approximation

$$w_k \approx \frac{1}{k} d^k$$

for  $k \geq 1$ . This results in

$$W(x) = \sum_{k=0}^{\infty} w_k x^k \approx 1 + \sum_{k=1}^{\infty} \frac{1}{k} d^k x^k = 1 - \log(1 - dx) \quad (2.25)$$

A sentence can be composed of only one word; The contribution to  $s_n$  is then  $w_n$ , and to  $S(x)$  is  $W(x)$ . It can be composed of two words; the contribution to  $s_n$  is then naively the convolution  $(w * w)_n = \sum_{m=0}^n w_m w_{n-m}$ , and to  $S(x)$  is  $W(x)^2$ , but we should remember that the order of the words is immaterial. As for large  $n$  the sentences composed of two identical words are negligible (the generating function of those is  $W(x^2)$ ), we should

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<sup>4</sup>Incidentally, the fact that  $w_p$  is integer provides a simple proof of Fermat's little theorem,  $d^p \equiv d \pmod{p}$ .

approximately divide by two. In general we approximate that the number of words is so large that generically no two will be identical in a sentence; if there are  $a$  words in a sentence, we should divide the naive approximation by  $a!$ . Consequently,

$$S(x) \approx \sum_{a=0}^{\infty} \frac{1}{a!} W(x)^a = e^{W(x)} \approx e \cdot \frac{1}{1-dx} = e \cdot \sum_{n=0}^{\infty} d^n x^n \quad (2.26)$$

so

$$s_n \approx e \cdot d^n \sim e^{(\log d)n} \quad (2.27)$$

and the Hagedorn temperature, in units of  $\omega$ , is indeed  $\frac{1}{\log d}$  as claimed.

Note that if

$$x = e^{-\beta\omega} \quad (2.28)$$

then

$$Z(\beta) \equiv S(x) = \sum_{n=0}^{\infty} s_n e^{-\beta n \omega} \quad (2.29)$$

is exactly the partition function, so in light of (2.26) we get

$$Z(\beta) \approx e \cdot \frac{1}{1 - d e^{-\beta\omega}} \quad (2.30)$$

The partition function clearly diverges at the inverse Hagedorn temperature  $\beta_H$  such that

$$d \cdot e^{-\beta_H \omega} = 1 \quad (2.31)$$

which gives indeed  $\beta_H = \frac{\log d}{\omega}$ .

In fact, a more careful use of Polya's theorem can give an exact expression [11] for  $Z(\beta)$  or  $S(x)$ , but this is irrelevant for our purposes.

## 2.2 Extensions

### 2.2.1 The Traceless Case

The matrices we have encountered thus far were assumed to be Hermitian. They can equivalently be characterized as matrices in the adjoint representation of the  $U(N)$  algebra. It is natural to explore the variant where the matrices are in the adjoint of  $SU(N)$  (we will denote the various magnitudes for the  $SU(N)$  version with a tilde). We will now show that our results are essentially unchanged if we have the gauge group  $SU(N)$  instead of  $U(N)$ .

A matrix in the adjoint representation of  $SU(N)$  is Hermitian, but with the extra constraint of being traceless. In particular  $\text{Tr } a^\dagger = 0$  and so there are no single-letter words:

$$\tilde{w}_1 = 0 \quad (2.32)$$

In the  $d = 1$  case, therefore, the partitions of  $n$  should not contain ones. As any partition without ones of any  $m \leq n$  can be uniquely completed, with the addition of  $n - m$  ones, to a partition of  $n$ , and vice versa, obviously

$$s_n = \sum_{m=0}^n \tilde{s}_m \quad (2.33)$$

and so

$$\tilde{s}_n = s_n - s_{n-1} \approx \frac{\partial}{\partial n} s_n \sim \frac{1}{\sqrt{n}} s_n \quad (2.34)$$

We see that the exponential behaviour, and hence the Hagedorn temperature, remains as in the  $U(N)$  case, and the change is only in the irrelevant polynomial prefactor. All this can also be seen from the fact that the relevant partition function is

$$\tilde{P}(x) = \prod_{i=2}^{\infty} (1 - x^i)^{-1} = (1 - x)P(x) \quad (2.35)$$

In the  $d > 1$  case, (2.32) implies

$$\tilde{W}(x) = W(x) - dx \quad (2.36)$$

and

$$\begin{aligned} \tilde{S}(x) &\approx e^{\tilde{W}(x)} = e^{-dx} e^{W(x)} \approx e^{-dx} S(x) \\ &\approx e(1 - dx + \frac{1}{2}d^2x^2 - \frac{1}{3!}d^3x^3 \pm \dots)(1 + dx + d^2x^2 + d^3x^3 + \dots) \end{aligned} \quad (2.37)$$

so the coefficient of  $x^n$  is now

$$s_n \approx e d^n (1 - 1 + \frac{1}{2} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!})$$

but the last factor is approximately  $e^{-1}$ , so

$$s_n \approx d^n \quad (2.38)$$

We get the same Hagedorn temperature as in the  $U(N)$  case, the only change being in an irrelevant constant factor.

### 2.2.2 Fermions

Fermionic matrix degrees of freedom behave quite similarly to bosonic ones of the same mass. We can form the fermionic creation operators  $b_i^\dagger$ . Each entry of such a matrix is a Grassmannian number which squares to zero; however, there are so many of those, (i.e.  $N^2$ ) that when forming words of finite length, none vanishes, and more generally, the words remain independent. However, each word squares to zero, so we can not use the same word more than once in a sentence.

For the  $d = 1$  case, this means that the partitions of  $n$  should be to distinct numbers. The number of such partitions,  $q(n)$ , behaves also as

$$q(n) \sim e^{c'\sqrt{n}} \quad (2.39)$$

when now  $c' = \sqrt{\frac{1}{3}}\pi$ . The corresponding generating function for the sentences is easily seen to be<sup>5</sup>

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<sup>5</sup>Incidentally, the number  $q(n)$  of partitions of  $n$  into distinct parts is equal to the the number of partitions into odd parts; this is an amusing exercise in generating functions, as formally

$$\prod_{i=1}^{\infty} (1 + x^i) = \prod_{j=1}^{\infty} (1 - x^{2j-1})^{-1}. \quad (2.40)$$

$$S(x) = Q(x) \equiv \sum_{n=0}^{\infty} q(n)x^n = \prod_{i=1}^{\infty} (1 + x^i) \quad (2.41)$$

For the  $d > 1$  case, we should again take into account the restriction that no two equal words appear inside a sentence. However, we have already argued that such cases are rare and used the approximation that they are negligible, when we argued that the “ $a$  words in a sentence” term  $W(x)^a$  should be divided by  $a!$  (so that  $S(x)$  is the exponent of  $W(x)$ ). We conclude that the Hagedorn temperature of the  $d > 1$  fermionic matrix model is equal to its bosonic counterpart.

### 2.2.3 Different Masses

When there are several matrices with different masses, there is obviously still a Hagedorn density of states, with the Hagedorn temperature between two bounds: the Hagedorn temperature of the same number of matrices but all with the same mass, which is either the minimal or maximal one. However, it is quite easy to generalize (2.31) to get an exact result.

**Claim.** *For a model with  $d > 1$  matrices, bosonic or fermionic, of masses  $\omega_i > 0$ ,  $i = 1 \dots d$ , the inverse Hagedorn temperature  $\beta_H$  is given by the solution of*

$$\sum_{i=1}^d e^{-\beta_H \omega_i} = 1 \quad (2.42)$$

To see that, we first pick a mass scale  $\omega$  and write

$$\omega_i = \omega \nu_i \quad (2.43)$$

The idea of the proof in the equal masses case carries over when we define

$$F(x) = \sum_{i=1}^d x^{\nu_i} \quad (2.44)$$

as a generalization of  $dx$ . We still have

$$W(x) \approx 1 + \sum_{k=1}^{\infty} \frac{1}{k} F(x)^k = 1 - \log(1 - F(x)) \quad (2.45)$$

and

$$S(x) \approx e^{W(x)} \approx e \cdot \frac{1}{1 - F(x)} \quad (2.46)$$

Where now  $s_n$  should be interpreted as the number of states with energy  $n\omega$  and not as the number of sentences of length  $n$ ; indeed,  $n$  doesn't even have to be integer.

Clearly,  $S(x)$  diverges at

$$x_H \equiv e^{-\beta_H \omega} \quad (2.47)$$

where

$$F(x_H) = 1 \quad (2.48)$$

Still using (2.28), we get

$$Z(\beta) \approx e \cdot \frac{1}{1 - \sum_{i=1}^d e^{-\beta w_i}} \quad (2.49)$$

which diverges at the claimed inverse Hagedorn temperature. As  $F$  is a monotone decreasing function of  $\beta$ , where  $F = d > 1$  for  $\beta = 0$  and  $F = 0 < 1$  for  $\beta = \infty$ , there is exactly one root of equation (2.42).

If  $\omega_i = 0$  for some  $i$ , then there is an infinite number of zero modes, the Hagedorn temperature is formally zero, and we see that the relation (2.42) continues to hold formally.

It is also easy to see that this reasoning, properly interpreted, does not rely upon the number of matrices being finite.

#### 2.2.4 Absence of Prefactor

We would also like to refine our analysis and find any energy power prefactor multiplying the leading exponential in the density of states. Stated otherwise, assuming that the density of states behaves as

$$\rho(E) \sim C E^\alpha e^{\beta_H E} \quad (2.50)$$

we wish to find  $\alpha$ .

**Claim.** *There is no power-like prefactor in matrix models, i.e.  $\alpha = 0$ .*

In order to see this, assume (2.50), and compute the partition function,

$$Z(\beta) = \int \rho(E) e^{-\beta E} dE = \int_0^\infty C E^\alpha e^{-(\beta - \beta_H)E} dE = \frac{C \Gamma(\alpha + 1)}{(\beta - \beta_H)^{\alpha + 1}} \quad (2.51)$$

or equivalently by (2.28,2.29,2.47),

$$S(x) = \frac{C \Gamma(\alpha + 1) \omega^{\alpha + 1}}{(\log(x_H/x))^{\alpha + 1}} \quad (2.52)$$

It is easy to see that

$$\frac{S(x) S''(x)}{S'(x)^2} = \frac{\alpha + 2 - \log(x_H/x)}{\alpha + 1} \quad (2.53)$$

so that  $\alpha$  can be extracted from the behaviour of the latter expression at the divergence,

$$\left. \frac{S(x) S''(x)}{S'(x)^2} \right|_{x=x_H} = \frac{\alpha + 2}{\alpha + 1} \quad (2.54)$$

On the other hand, it is immediate from (2.46) that

$$\frac{S(x) S''(x)}{S'(x)^2} = \frac{2F'(x)^2 - (F(x) - 1)F''(x)}{F'(x)^2} \quad (2.55)$$

and therefore, from (2.48),

$$\left. \frac{S(x) S''(x)}{S'(x)^2} \right|_{x=x_H} = 2 \quad (2.56)$$

Equating (2.54) and (2.56) we get indeed that  $\alpha = 0$  as claimed.

Essentially, this behaviour is dictated by the fact that  $F(x)$ , a monotonous function, has a non zero derivative at  $x = x_H$ .

### 2.3 Hagedorn Behaviour in Free Second Quantized Theories

Let us study a free quantum theory of harmonic oscillators of frequencies  $\omega_i > 0$ . Their number may be finite or infinite, and they might be bosonic or fermionic. The corresponding Hamiltonian, after discarding the zero-energy, is  $H = \sum_i \omega_i a_i^\dagger a_i$ . The Fock space states of the second quantized theory (the “sentences”) are generated directly by applying the creation operators (the “letters”) on the vacuum: as the creation operators are not matrices, and no gauge invariance condition is imposed, there is no need for “words” in this context.

**Claim.** *The inverse Hagedorn temperature  $\beta_H$  in this model is given by the solution of*

$$\sum_i e^{-\beta_H \omega_i} = \infty \quad (2.57)$$

*More accurately, it is the maximal value of  $\beta$  where the left hand side diverges.*

Note that this implies that the Hagedorn temperature for a system with a finite number of oscillators is always infinite, so there is no Hagedorn transition (for that, it is essential that there are no zero modes  $\omega_i = 0$ ). Even for systems with an infinite number of oscillators the Hagedorn temperature might be infinite. It is indeed infinite in the sensible cases where the model comes from a field theory in a finite volume.

To understand the claim, we will begin with the example of  $d$  bosonic oscillators of equal frequency  $\omega$ . As in the the corresponding matrix case of subsection 2.1.2,  $s_n$  counts the number of sentences of length  $n$  made out of  $d$  letters. The order of the letters does not matter, as the creation operators commute, and therefore  $s_n = \binom{n+d-1}{d-1}$  (think of arranging  $n$  circles and  $d-1$  dividers in a line). As the density of states  $s_n \approx \text{const} \cdot n^{d-1}$  behaves polynomially and not exponentially, there is no Hagedorn behaviour.

This can also be easily seen from the generating function:

$$S(x) = \sum_{n=0}^{\infty} \binom{n+d-1}{d-1} x^n = \frac{1}{(1-x)^d} = \prod_{i=1}^d (1 + x + x^2 + x^3 + \dots) \quad (2.58)$$

The term  $x^j$  in the  $i$ -th multiplicand represents  $j$  excitations of the  $i$ -th oscillator.

The generalization to different frequencies is immediate: using (2.28,2.43), a bosonic oscillator  $a^\dagger$  contributes a factor of  $\frac{1}{1-x^\nu}$ , while a fermionic one can be excited at most once, as  $(b^\dagger)^2 = 0$ , and so contributes a factor of  $1 + x^\nu$ . Accordingly,

$$S(x) = \prod_{\text{Bosons}} \frac{1}{1-x^{\nu_i}} \cdot \prod_{\text{Fermions}} (1+x^{\nu_i}) \quad (2.59)$$

Now, take the logarithm:

$$\log S(x) = \sum_{\text{Bosons}} -\log(1-x^{\nu_i}) + \sum_{\text{Fermions}} \log(1+x^{\nu_i}) \quad (2.60)$$

If there is a subsequence of the  $\nu_i$  going to zero, then  $\log S(x)$  diverges for all  $x > 0$ , so by (2.28),  $\beta_H = \infty$ ; this is clearly true also from (2.57). Assume, on the other hand, that

$\nu^* \equiv \inf \nu_i > 0$ . Then for  $1 > x > 0$  we have  $x^{\nu_i} \leq x^{\nu^*} < 1$ , so there is some  $\eta < 1$  such that  $(1 - \eta)x^{\nu_i} \leq -\log(1 - x^{\nu_i})$ ,  $\log(1 + x^{\nu_i}) \leq (1 + \eta)x^{\nu_i}$ . Consequently, (2.60) and the right hand of (2.57) converge or diverge together. This concludes the proof of the claim.

As an example, let us take a classical (non-relativistic) string tied in its two ends and free to oscillate in one transverse dimension. The string's frequencies are harmonics of some natural frequency,  $\omega_i = i \cdot \omega$  for  $i = 1, 2, 3, \dots$ . The number of states of energy  $n \cdot \omega$  is again  $p(n)$ , the number of partitions of  $n$ , which is subexponential (2.22), so there is no Hagedorn behaviour.  $S(x) = P(x)$  is given by (2.21) that has a radius of convergence one around the origin. In this case, The right hand side of (2.57),  $\sum_i e^{-\beta\omega_i} = \sum_i x^i = \frac{1}{1-x}$  has indeed the same radius of convergence. If the string lives in  $D$  spacetime dimensions, there are  $D - 2$  transverse directions, so  $S(x) = P(x)^{D-2}$ , which still displays no Hagedorn behaviour.

The lack of Hagedorn behaviour is generic in field theories. In the free case in a finite volume, the number of oscillators grows like the phase space, or as a power of the energy. The result (2.57) implies that we should look at expressions of the form  $\sum_{i=0}^{\infty} i^d x^i \sim \frac{1}{(1-x)^{d+1}}$ , as was shown, in a different context, in (2.58). The divergence is still at  $x_H = 1$ , corresponding to  $\beta_H = 0$  or  $T_H = \infty$ .

### 3. The BMN Matrix Model

As mentioned in the introduction, this model arises from the study of a certain M-theory plane wave which is the Penrose limit of the  $\text{AdS}_4 \times S^7$  or  $\text{AdS}_7 \times S^4$  solutions. An interacting matrix model arises from the DLCQ description of this background when there are  $N$  units of momentum on the light-like circle [4].

There are  $11 - 2 = 9$  coordinates apart from the light cone ones, which break into two groups: three coordinates which are equivalent under an  $SO(3)$  rotation, and the other six, equivalent under  $SO(6)$ . Those coordinates give rise to nine bosonic matrices: three of mass  $\frac{1}{3}\omega$  for the appropriate  $\omega$ , and six of mass  $\frac{1}{6}\omega$ . The model is supersymmetric and correspondingly has eight fermionic matrices, all of mass  $\frac{1}{4}\omega$ .

The various classical solutions of the matrix model are given by vanishing  $SO(6)$  (and fermionic) matrices, and by the  $SO(3)$  matrices being a dimension  $N$  representation of  $SU(2)$ . Taking the small 't Hooft limit makes the theory free around the classical solution.

#### 3.1 A Single Five-Brane

A single spherical five-brane solution is represented by the classical solution with the  $SO(3)$  matrices vanishing (or being in the trivial representation) [10]. This vacuum manifestly respects the  $U(N)$  gauge symmetry of the model. The oscillators are then just given by the bosonic and fermionic  $U(N)$  matrices described above.

According to our previous result (2.42), the inverse Hagedorn temperature is given by the solution of

$$3e^{-\beta\omega/3} + 8e^{-\beta\omega/4} + 6e^{-\beta\omega/6} = 1 \quad (3.1)$$



or, if  $\hat{x} = e^{-\beta\omega/12}$ , by the solution  $0 \leq \hat{x} \leq 1$  of

$$(\hat{x} + 1)^3(\hat{x} - \frac{1}{3}) = 3\hat{x}^4 + 8\hat{x}^3 + 6\hat{x}^2 - 1 = 0 \quad (3.2)$$

that is, by  $\hat{x} = \frac{1}{3}$ , so  $\beta_H = -\log \hat{x} \cdot \frac{12}{\omega} = \frac{12 \log 3}{\omega} \approx \frac{13.1833}{\omega}$  or  $T_H \approx 0.0758533\omega$ .

### 3.2 Multiple Five-Branes

A solution of  $k$  coincident spherical five-branes corresponds to the matrix model classical vacuum with the  $SO(3)$  matrices being  $n$  copies of the  $k$  dimensional representation, where  $n \equiv N/k$  is taken to infinity, with  $k$  fixed. The  $SO(3)$  matrices having a vacuum expectation value results in the Higgsing of the gauge group to  $U(n)$ . Each  $U(N)$  matrix is decomposed into  $k^2$  different  $U(n)$  matrix oscillators, which arrange themselves into multiplets of  $SO(3)$  spins  $j$ , integer for the bosonic oscillators and half integer for the fermionic ones. As  $k$  grows, more and more oscillators are added, as the maximal  $j$  grows by one for each oscillator type. Essentially, each oscillator becomes a tower of oscillators, with the original  $k = 1$  oscillator at the bottom, and two new towers begin at  $k = 2$ . As a consequence, the Hagedorn temperature is lowered. We will see, however, that the change is very small even in the large  $k$  limit, because the masses of the oscillators grow linearly in  $j$  (their degeneracy does also, of course, but this is much less important).

The spectrum of oscillators in this case was computed in [5, 6], and is given (in units of  $\omega$ ) in table 1, where we also give the labels given there to the different oscillator towers.

Type	Label	Mass	Spins	$SO(6) \times SO(3)$
Bosonic $SO(6)$	$x$	$\frac{1}{6} + \frac{j}{3}$	$0 \leq j \leq k-1$	$(6, 2j+1)$
Bosonic $SO(3)$	$\alpha$	$\frac{1}{3} + \frac{j}{3}$	$0 \leq j \leq k-2$	$(1, 2j+1)$
	$\beta$	$\frac{j}{3}$	$1 \leq j \leq k$	$(1, 2j+1)$
Fermionic	$\chi$	$\frac{1}{4} + \frac{j}{3}$	$\frac{1}{2} \leq j \leq k - \frac{3}{2}$	$(\bar{4}, 2j+1)$
	$\eta$	$\frac{1}{12} + \frac{j}{3}$	$\frac{1}{2} \leq j \leq k - \frac{1}{2}$	$(4, 2j+1)$

**Table 1:** Oscillator masses for  $k$  coincident five-branes (adapted from [5])

The contribution of the oscillators labeled by “ $x$ ” is

$$\sum_{j=0}^{k-1} 6(2j+1)\hat{x}^{4j+2} = \frac{6(\hat{x}^2 + \hat{x}^6 - (2k+1)\hat{x}^{4k+2} + (2k-1)\hat{x}^{4k+6})}{(1-\hat{x}^4)^2} \quad (3.3)$$

which converges in the large  $k$  limit (as  $\hat{x} < 1$ ) to

$$\frac{6(\hat{x}^2 + \hat{x}^6)}{(1-\hat{x}^4)^2} = \frac{6\hat{x}^2(1+\hat{x}^4)}{(1-\hat{x}^4)^2} \quad (3.4)$$

The computations for the other kinds of oscillators are similar, and we will only display the large  $k$  limit result. The “ $\alpha$ ” oscillators contribute  $\hat{x}^4(1+\hat{x}^4)/(1-\hat{x}^4)^2$ , the “ $\beta$ ” oscillators

$(3\hat{x}^4 - \hat{x}^8)/(1 - \hat{x}^4)^2$ , the “ $\chi$ ” oscillators  $8\hat{x}^5/(1 - \hat{x}^4)^2$  and the “ $\eta$ ” oscillators  $8\hat{x}^3/(1 - \hat{x}^4)^2$ . The sum of all those contributions should equal one, so after multiplying by  $(1 - \hat{x}^4)^2$  we get

$$(\hat{x} + 1)^4(\hat{x}^4 - 4\hat{x}^3 + 4\hat{x}^2 - 4\hat{x} + 1) = 0 \quad (3.5)$$

having a unique real root between zero and one,

$$\hat{x} = 1 + \frac{1}{\sqrt{2}} - \sqrt{\sqrt{2} + \frac{1}{2}} \approx 0.323556 \quad (3.6)$$

so  $\beta_H = -\log \hat{x} \cdot \frac{12}{\omega} \approx \frac{13.5406}{\omega}$  and  $T_H \approx 0.0738519\omega$ . Note how close these values are to those of the single five-brane ( $k = 1$ ) case.

### 3.3 Membranes

The matrix model vacuum in which the  $SO(3)$  matrices are in the  $N$  dimensional irreducible representation corresponds to a spherical membrane. The vacuum with  $k$  copies of the  $n$  dimensional irreducible representation, with  $n \equiv N/k \rightarrow \infty$ , and  $k$  fixed, corresponds to  $k$  coincident spherical membranes [4, 5]. The spectrum of excitations above those vacua was computed in [5, 6] and is given in table 2 (again in units of  $\omega$ ).

Type	Label	Mass	Spins	$SO(6) \times SO(3)$
Bosonic $SO(6)$	$x$	$\frac{1}{6} + \frac{j}{3}$	$0 \leq j \leq n - 1$	$(6, 2j + 1)$
Bosonic $SO(3)$	$\alpha$	$\frac{1}{3} + \frac{j}{3}$	$0 \leq j \leq n - 2$	$(1, 2j + 1)$
	$\beta$	$\frac{j}{3}$	$1 \leq j \leq n$	$(1, 2j + 1)$
Fermionic	$\chi$	$\frac{1}{4} + \frac{j}{3}$	$\frac{1}{2} \leq j \leq n - \frac{3}{2}$	$(\bar{4}, 2j + 1)$
	$\eta$	$\frac{1}{12} + \frac{j}{3}$	$\frac{1}{2} \leq j \leq n - \frac{1}{2}$	$(4, 2j + 1)$

**Table 2:** Oscillator masses for  $k$  coincident membranes (adapted from [5])

In the single membrane case, the gauge group is completely Higgsed, and the oscillators are not matrices anymore. Although there is an infinite number of such oscillators as  $N \rightarrow \infty$ , their density is just linear (the mass grows as the spin  $j$  while the degeneracy is proportional to  $2j + 1$ ), and as explained in subsection 2.3, the Hagedorn temperature is infinite.

In the case of  $k$  coincident membranes, the gauge group is Higgsed down to  $U(k)$ , and the oscillators are  $k$  dimensional matrices. Each oscillator, then, is comprised of  $k^2$  entries, a fixed finite number, and viewing those as the basic oscillators, even disregarding the gauge invariance condition, shows that the density of oscillator masses is still insufficient to produce a Hagedorn behaviour.

## 4. Thermodynamics of the Matrix Model

In the previous section, we demonstrated that the spectrum of the matrix model has a Hagedorn density of states in the large  $N$  limit and that we should expect a phase transition

at some temperature of order the parameter  $\omega$ . In this Section, we will demonstrate that this is indeed the case using a direct analysis of the thermodynamics of the matrix model. We will be interested in the thermodynamic partition function, where we take the Hamiltonian,  $H$ , for the model corresponding to (1.2) and form the partition function

$$Z[N, R, \beta] = \text{Tr} e^{-\beta H} = e^{-\beta F[N, R, \beta]}$$

where  $\beta = 1/T$  is the inverse temperature and  $F[N, R, \beta]$  is the free energy. Before we discuss this, we shall again illustrate the method by using a simple toy model.

#### 4.1 The Toy Model

Consider again the toy model of section 2. The thermodynamic partition function has a functional integral representation

$$Z = \int [dA][dX_j] e^{-\int_0^\beta d\tau \frac{1}{2} \sum_j \text{Tr}((DX_j)^2 + \omega^2 X_j^2)} \quad (4.1)$$

where the fields have periodic boundary conditions in Euclidean time

$$A(\tau + \beta) = A(\tau) \quad , \quad X_j(\tau + \beta) = X_j(\tau) \quad (4.2)$$

To proceed, we must fix a gauge. The periodic boundary condition prevents us from choosing the  $A = 0$  gauge. The best we can do is to fix a gauge where the gauge field is static and diagonal,

$$\frac{d}{d\tau} A(\tau) = 0 \quad , \quad A^{ab} = \delta^{ab} A_a \quad (4.3)$$

In that case,

$$(DX_j)^{ab} = \frac{d}{d\tau} X_j^{ab} + i(A_a - A_b) X_j^{ab} \quad (4.4)$$

The Faddeev-Popov determinant for fixing this gauge is computed in the Appendix. It turns out to be similar to the Vandermonde determinant which appears in the measure of unitary matrix integrals,

$$\det D \det' \frac{d}{d\tau} = \prod_{a \neq b} \sin \frac{\beta}{2} |A_a - A_b| \quad (4.5)$$

where  $\det'$  is a determinant restricted to the space of non-constant modes. Then, the integral over the oscillator fields in (4.1) is Gaussian and can be performed explicitly

$$Z = \int \prod_{a=1}^N dA_a \prod_{a \neq b} \sin \frac{\beta}{2} |A_a - A_b| \prod_{a,b} \det^{-\frac{d}{2}} \left( - \left( \frac{d}{d\tau} + i(A_a - A_b) \right)^2 + \omega^2 \right) \quad (4.6)$$

The determinant can be evaluated explicitly (see the Appendix, also ref. [22]). This model with  $d = 1$ , and a different critical behavior from the one which we shall be discussing in the following, has been studied extensively in the context of lower dimensional string theories. For example, see refs. [23],[24].

We obtain

$$Z = \int \prod_{a=1}^N dA_a \prod_{a \neq b} \sin \frac{\beta}{2} |A_a - A_b| \prod_{a,b} \left( \frac{1}{\sinh \left( \frac{\beta}{2} (\omega + i(A_a - A_b)) \right)} \right)^d \quad (4.7)$$

The effective action for the eigenvalues is then

$$S_{\text{eff}} = \sum_{a \neq b} \left\{ d \ln \left( \sinh \frac{\beta}{2} (\omega + i(A_a - A_b)) \right) - \ln \sin \frac{\beta}{2} |A_a - A_b| \right\} \quad (4.8)$$

We have dropped an  $A$ -independent ground state energy. There are now  $N$  remaining degrees of freedom, the variables  $A_a$ , and because there are two  $N$ -fold summations, the action is of order  $N^2$ . For this reason, in the large  $N$  limit the integration in (4.6) can be done in the saddle point approximation. In that case there is a well-defined expansion of the integral as a power series in  $1/N^2$ . To begin, we must find the saddle point by minimizing the effective action. The equation for the minimum, for each  $a$ , is

$$\sum_{b \neq a} d \frac{\sin \beta(A_a - A_b)}{\cosh \beta\omega - \cos \beta(A_a - A_b)} = \sum_{b \neq a} \cot \frac{\beta}{2} (A_a - A_b) \quad (4.9)$$

It is useful to introduce the eigenvalue density,

$$\rho(\theta) = \frac{1}{N} \sum_{a=1}^N \delta(\theta - \beta A_a) \quad (4.10)$$

and to assume it is a smooth function in the large  $N$  limit. Without loss of generality,  $-\pi \leq \theta \equiv \beta A \leq \pi$ , and the density is normalized so that

$$\int_{-\pi}^{\pi} d\theta \rho(\theta) = 1 \quad (4.11)$$

Obviously, the density is constrained to be positive,  $\rho(\theta) \geq 0$ . Now, the action can be written as

$$S_{\text{eff}} = N^2 \int d\theta d\theta' \rho(\theta) \rho(\theta') \left\{ d \ln \left( \sinh \left( \frac{\beta\omega}{2} + \frac{i}{2}(\theta - \theta') \right) \right) - \ln \left( \sin \frac{1}{2} |\theta - \theta'| \right) \right\} \quad (4.12)$$

We see there is a sort of competition between the eigenvalue repulsion of the second term, coming from the Vandermonde, and the eigenvalue attraction of the first term, coming from the action. The attraction is more pronounced in the high temperature regime,  $\beta \rightarrow 0$ .

The equation for the eigenvalue density is given by

$$\int d\theta' \rho(\theta') d \frac{\sin(\theta - \theta')}{\cosh \beta\omega - \cos(\theta - \theta')} = \int d\theta' \rho(\theta') \cot \frac{1}{2} (\theta - \theta') \quad (4.13)$$

where it is understood that the principal value around the singularity should be taken in the right hand side integral. Note that this equation should be satisfied only in the support of  $\rho(\theta)$ , that is, for angles  $\theta$  such that  $\rho(\theta) > 0$ . This equation should be solved together with the normalization condition (4.11) and the positivity constraint.

First, we observe that the constant eigenvalue density

$$\rho_0 = \frac{1}{2\pi} \quad (4.14)$$

is always a solution of (4.13) where both sides of the equation vanish. However, this solution is stable only for a range of values of  $\beta$ . To see this, we Fourier expand inside (4.12),<sup>6</sup>

$$S_{\text{eff}} = N^2 \int d\theta d\theta' \rho(\theta) \rho(\theta') \sum_{n=1}^{\infty} \frac{1 - de^{-n\beta\omega}}{n} \cos(n(\theta - \theta')) \quad (4.15)$$

From this, we can see that if  $\beta$  is large enough, then the phase where the density is constant (so that the Fourier modes vanish,  $\int d\theta \rho(\theta) e^{in\theta} = 0$  for  $n \neq 0$ ) is stable. As the temperature is raised, and  $\beta$  is decreased, the first instability sets in at  $\beta = \beta_H = 1/T_H$  where

$$T_H = \frac{\omega}{\ln d} \quad (4.16)$$

This coincides with the Hagedorn temperature in (2.7) which we found by estimating the high energy density of states in section 2.

Note that the approximations of section 2 amount to keeping the  $n = 1$  term only in (4.15), and that the arguments there are invalid in the deconfined phase above the Hagedorn temperature. However, the result of section 2 correctly gives the free energy below the Hagedorn temperature. All that (4.15) tells us about the low temperature phase is that the free energy there is zero in terms of  $N^2$ ; indeed, it is  $O(N^0)$  in the confined phase, but more elaborate computations are needed in order to reproduce this from the matrix theory perspective.

The straightforward generalization of this model where we have different frequencies for the bosonic oscillators and possibly some fermionic oscillators gives the general equation for the critical temperature of the matrix model,

$$\sum_i e^{-\beta_H \omega_i} = 1 \quad (4.17)$$

This also coincides with the equation for the Hagedorn temperature which was found in section 2.

The phase transition that we have identified is the usual Gross-Witten [25] transition of unitary matrix models. In that phase transition the distribution of eigenvalues of the unitary matrix rearranges itself from one which has compact support near the identity matrix to one which is distributed on the whole unit circle.

Above the phase transition the eigenvalue distribution is not uniform. Without loss of generality, we may assume it is an even function concentrated around  $\theta = 0$ . The distribution can be approximated by the semi-circle one,

$$\rho(\theta) = \frac{2}{\pi\xi} \cos \frac{\theta}{2} \sqrt{\frac{\xi}{2} - \sin^2 \frac{\theta}{2}} \quad (4.18)$$

---

<sup>6</sup>Here, we use the identity

$$\ln \left( \sin \frac{\theta}{2} \right) = - \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} + \text{const.}$$

This distribution has support in the region  $|\sin \frac{\theta}{2}| \leq \sqrt{\frac{\xi}{2}}$  and the parameter  $\xi$  is restricted to the range  $\xi < 2$ . This approximation is exact when we truncate the low temperature expansion of the matrix contribution to the effective action at its first nontrivial,  $\rho(\theta)$  dependent, term

$$d \ln(1 - e^{-\beta\omega - i(\theta - \theta')}) \approx -de^{-\beta\omega} e^{-i(\theta - \theta')} \quad (4.19)$$

and the saddle point equation is approximately

$$\int d\theta' \rho(\theta') \left( 2de^{-\beta\omega} \sin(\theta - \theta') - \cot \frac{1}{2}(\theta - \theta') \right) = 0 \quad (4.20)$$

This equation is solved by (4.18) when <sup>7</sup>

$$\xi = 2 \left( 1 - \sqrt{1 - 1/de^{-\beta\omega}} \right) \quad (4.21)$$

This solution exists only when  $de^{-\beta\omega} \geq 1$ , that is when the temperature is greater than the Hagedorn temperature. In this region  $\xi < 2$  and the semi-circle distribution is well-defined. Note, however, that the expansion parameter in the approximation (4.19),  $e^{-\beta\omega}$ , is at least  $e^{-\beta_H\omega} = \frac{1}{d}$ .

This approximation is not too bad, however, even in the high temperature limit. For  $\beta = 0$  we get a narrow distribution, as  $\xi \sim \frac{1}{d}$ . It is easy, though, to solve exactly for the density in this limit. Equation (4.13) reduces, for  $d > 1$ , to

$$\int d\theta' \rho(\theta') \cot \frac{1}{2}(\theta - \theta') = 0$$

Because of the implicit principle value we immediately see that in this limit,  $\rho(\theta) = \delta(\theta)$  is a solution (with support only at  $\theta = 0$ ); the eigenvalue attraction totally wins over.

## 4.2 Confinement-Deconfinement Transition

The matrix model phase transition that we have found is characteristic of a deconfining phase transition in a gauge theory which is heated to a sufficiently high temperature. The matrix quantum mechanics has the gauge symmetry (2.2) where, since all variables transform under the adjoint representation, the unitary matrices are periodic up to an element of the center of the group,

$$U(\tau + \beta) = zU(\tau) \quad (4.22)$$

---

<sup>7</sup>To see this, we need the integrals

$$\int d\theta' \rho(\theta') \cot \frac{1}{2}(\theta - \theta') = \frac{2}{\xi} \sin \theta$$

where again the principal value is implied, and

$$\int d\theta \rho(\theta) \cos \theta = \left( 1 - \frac{\xi}{4} \right)$$

and where  $z$  is an element of the center of the group. If the gauge group is  $U(N)$ , the center is  $U(1)$ . If it is  $SU(N)$ , the center is  $Z_N$ , the multiplicative group of the  $N$ 'th roots of unity.

No local operators are sensitive to the center element in (4.22). However, there is a nonlocal operator which transforms under the center, the Polyakov loop operator

$$P = \text{Tr } \mathcal{P} \exp \left( i \int_0^\beta d\tau A(\tau) \right) \quad (4.23)$$

where  $\mathcal{P}$  denotes path ordering. Under a gauge transform with boundary condition (4.22),

$$P \rightarrow zP \quad (4.24)$$

The center symmetry is often treated as a global symmetry of the finite temperature gauge theory. As a global symmetry one can ask the question as to whether or not it is spontaneously broken. An order parameter is the expectation value of the Polyakov loop,  $\langle P \rangle$ . If the center symmetry is unbroken, this expectation value must vanish. If it is broken, then  $\langle P \rangle$  can be non-zero.

The vanishing or non-vanishing of  $\langle P \rangle$  has an interpretation in terms of confinement. The expectation value,

$$\langle P \rangle = \frac{\int [dA][dX] e^{-\int_0^\beta d\tau \frac{1}{2} \text{Tr}((DX)^2 + \omega^2 X^2)} \text{Tr } \mathcal{P} e^{i \int_0^\beta d\tau A(\tau)}}{\int [dA][dX] e^{-\int_0^\beta d\tau \frac{1}{2} \text{Tr}((DX)^2 + \omega^2 X^2)}} \equiv \exp(-\beta(F - F_0)) \quad (4.25)$$

is the ratio of partition functions for the system where one classical quark source is inserted to the partition function in the absence of the source.  $F - F_0$  is interpreted as the difference of the free energies with and without the quark. When the expectation value vanishes, this is interpreted as taking an infinite amount of energy to insert the quark. Thus, the phase with unbroken center symmetry is confining. When the symmetry is broken, the free energy of the extra quark is finite and that phase is deconfined. Some examples of lower dimensional gauge theories where the expectation value can be computed appear in ref. [33, 34, 35]. In particular, the case of the non-abelian coulomb gas could be written as an effective unitary matrix model and it exhibited a phase transition very similar to the one that we have studied here [34, 35].

### 4.3 The Matrix Model

The variables in the action of the matrix model (1.2) can be rescaled so that it has the form

$$\begin{aligned} S = \left( \frac{\mu}{3R} \right)^3 \int_0^{\beta\mu/3} d\tau \text{Tr} & \left( \frac{1}{2} (DX^i)^2 + \psi^\dagger D\psi + \frac{1}{2} X^{a2} + \frac{1}{8} X^{a'2} + \frac{3}{4} \psi^\dagger \psi \right. \\ & \left. + i\epsilon^{abc} X^a X^b X^c + \psi^\dagger \sigma^a [X^a, \psi] \right. \\ & \left. - \frac{1}{2} \epsilon_{\alpha\beta} \psi^{\dagger\alpha I} g_{IJ}^{a'} [X^{a'}, \psi^{\dagger\beta J}] + \frac{1}{2} \epsilon_{\alpha\beta} \psi^{\alpha I} g_{IJ}^{a'\dagger} [X^{a'}, \psi^{\beta J}] - \frac{1}{4} [X^i, X^j]^2 \right) \end{aligned} \quad (4.26)$$

Here,  $\sigma^a$  are usual Pauli matrices and  $g_{IJ}^{a'}$  relate the  $SU(4)$  fundamentals to an  $SO(6)$  fundamental [5]. As discussed in the Appendix, rescaling of the coordinates does not affect the path integral measure.

We will consider the semiclassical expansion of this model about the classical vacuum corresponding to the single five-brane,

$$X_{\text{cl}} = 0$$

Expansion to one loop order around this solution gives the semi-classical partition function

$$Z \approx \prod_{a=1}^N \int_0^{2\pi} [dA_a] \frac{(\det_F(D + 3/4))^8 |\det_B(D)| |\det'_B(d/d\tau)|}{(\det_B(-D^2 + 1))^{3/2} (\det_B(-D^2 + 1/4))^3} \quad (4.27)$$

Where the differential operators in  $\det_B$  and  $\det_F$  have periodic and anti-periodic boundary conditions, respectively. We discuss how to evaluate the determinants explicitly in the Appendix. The result is

$$Z \approx \prod_{a=1}^N \int_0^{2\pi} [dA_a] \prod_{a \neq b} \frac{|\cosh \frac{1}{2} (\frac{\mu}{4T} + i(A_a - A_b))|^8 \sin \frac{1}{2} |A_a - A_b|}{|\sin \frac{1}{2} (\frac{\mu}{3T} + i(A_a - A_b))|^3 |\sin \frac{1}{2} (\frac{\mu}{6T} + i(A_a - A_b))|^6} \quad (4.28)$$

When  $N$  is finite, this integral is a smooth function of  $\mu/T$ . When  $N$  is put to infinity, it can have singularities and it typically has a phase transition at some critical value of  $\mu/T$ . In that limit, the action, which is the logarithm of the integrand, is of order  $N^2$ , whereas there are  $N$  degrees of freedom,  $A_a$ . In the infinite  $N$  limit, the integral can be evaluated by saddle point technique. When  $\mu/T$  is a large number, the integrand is dominated by the Faddeev-Popov term which tends to spread the eigenvalues apart, giving a uniform distribution of  $e^{iA_a}$  around the unit circle in the complex plane. When  $\mu/T$  is small, on the other hand, the terms in the denominator cancel the Faddeev-Popov term and the remainder of the integrand emphasizes configurations where the eigenvalues are close together. Because of the translation invariance of the potential, which is a result of the center symmetry of the action, the accumulation point of the eigenvalues is arbitrary and breaks the  $Z_N$  symmetry of the effective theory.

In order to evaluate the integral in the infinite  $n$  limit, we must find the minimum of the action

$$S_{\text{eff}} = \int d\theta \rho(\theta) d\theta' \rho(\theta') \sum_{n=1}^{\infty} \frac{1}{n} \left[ 1 + 8(-1)^n e^{-\mu n/4T} - 3e^{-\mu n/3T} - 6e^{-\mu n/6T} \right] e^{in(\theta - \theta')}$$

It is easy to see that, if the temperature is large enough, the potential favors a non-zero value of the Fourier transforms

$$\rho_n = \int_{-\pi}^{\pi} d\theta \rho(\theta) e^{in\theta}$$

In fact, the first one to condense is  $\rho_1$  ( $\rho_0 = 1$  is constrained by normalization). What keeps it finite is the constraint that  $\rho(\theta) \geq 0$  for all values of  $\theta$ .



The critical temperature for  $\rho_n$  satisfies the equation

$$1 + 8(-1)^n e^{-\mu n/4T_{\text{crit}}} - 3e^{-\mu n/3T_{\text{crit}}} - 6e^{-\mu n/6T_{\text{crit}}} = 0 \quad (4.29)$$

At the phase transition the free energy changes from a function which is zero in leading order to one which is non-zero. The phase transition is first order.

The reason why one can have a phase transition in such a low dimensional system (the only dimension is the one labeled by  $\theta$  which comes from the matrix index) is because the effective action is non-local, having infinite range interactions.

## 5. Discussion

### 5.1 M-Theory Perspective

The energy scale  $\omega$  of the excitations above the different vacua was recognized in [5] to be the scale  $\mu$  appearing in the M-theory plane wave background and in the matrix action (1.2). The higher order interactions are governed by the coupling

$$g_{\text{eff}} = \left( \frac{3R}{\mu} \right)^{\frac{3}{2}} \quad (5.1)$$

where we work in Planck length units,  $\ell_p = 1$ . Moreover, in the five-brane case, the energy shifts of the excitation masses from the free, quadratic, values are given by the 't Hooft coupling

$$\lambda \equiv g_{\text{eff}}^2 N = \left( \frac{3R}{\mu} \right)^3 N \quad (5.2)$$

so our results are valid only when  $\lambda$  is small,  $\lambda \ll 1$ . Obviously we need also to take  $N \rightarrow \infty$  for the 't Hooft limit in the matrix model. Furthermore, we need to keep  $\mu$  fixed in order to get a nontrivial Hagedorn temperature result. All this means that we take the limit in which the DLCQ radius vanishes,  $R \rightarrow 0$ . The brane light cone momentum,

$$p^+ = \frac{N}{R} \quad (5.3)$$

therefore diverges in this limit,  $p^+ \rightarrow \infty$ . The radius of the five-sphere on which the five-brane wraps can be computed in the small coupling regime [10]. Classically the  $SO(6)$  directions are zero, but quantum mechanically, at one loop order, they give

$$r_5 = \left\langle \frac{1}{N} \text{Tr} \left( X^{a'} \right)^2 \right\rangle^{1/2} = \left( \frac{18N^2}{\mu p^+} \right)^{1/2} = \left( \frac{18NR}{\mu} \right)^{1/2} = \left( 6\lambda^{1/3} N^{2/3} \right)^{1/2} \quad (5.4)$$

so it diverges if  $\lambda$  is kept fixed, but not necessarily if, additionally,  $\lambda$  is taken to zero in this limit.

The five-branes we have studied are therefore quite peculiar from the M-theory point of view. As solutions with a light-like time direction, they are in the infinite momentum frame, moving (almost) at the speed of light. They have large momentum on another light-like direction which is compact and small. This makes it difficult to compare them

directly with ordinary NS five-branes or M five-branes. The five-branes might be flat (with  $r_5$  large in Planck length units) or not, according to the exact limit taken.

We would have liked, of course, to explore the behaviour of the five-branes in the large  $N$  limit where  $\mu$  and  $p^+$  (and therefore  $r_5$ ) are fixed. This however implies  $R \rightarrow \infty$ , so  $g_{\text{eff}} \rightarrow \infty$  and, a fortiori,  $\lambda \rightarrow \infty$ , where our approximations are invalid.

Anticipating the results of the next subsection, let us look at the five-sphere radius in terms of the length scale  $\frac{\mu}{R}$  (which is the typical (classical) length scale of our matrix model, as can be seen from the rescaling done in (4.26)). We obtain

$$r_5^2 \cdot \left(\frac{\mu}{R}\right)^{-2} = \frac{2}{3}\lambda \quad (5.5)$$

depends only on the 't Hooft coupling and is directly proportional to it. The five-brane radius  $r_{5,\text{cl}}$  can also be derived from the the classical five-brane action [10],

$$r_{5,\text{cl}}^4 = \frac{1}{6} \frac{\mu N}{R} \quad (5.6)$$

This calculation is valid for *large* values of  $\lambda$ . measuring again in terms of  $\frac{\mu}{R}$  we get

$$r_{5,\text{cl}}^2 \cdot \left(\frac{\mu}{R}\right)^{-2} = \frac{1}{9\sqrt{2}}\sqrt{\lambda} \quad (5.7)$$

This again depends only on  $\lambda$ , but has a different power dependence. It is interesting to note that for the crossover region  $\lambda \approx 1$ ,  $r_5$  and  $r_{5,\text{cl}}$  are of the same order. It is natural to assume that there is a smooth transition between the small and large 't Hooft coupling results.

When comparing a single five-brane and a stack of  $k$  of those, we would like to deal with the same radius,  $r_5$ , and momentum per brane,  $p^+$ ; as we have seen, both must be large. The equations (5.3,5.4) still apply, provided we replace  $N$  by  $n = N/k$  in them;  $n$  is, so to speak, the amount of “ $N$  per brane”. We therefore get that  $\mu$  is fixed, and hence, as we have seen in this section, the Hagedorn temperature is fixed up to a very small numerical factor. As we need  $n$  fixed as we vary  $k$ , the total  $N$  is proportional to  $k$ , so we are not directly comparing different vacua of the matrix model with the same parameters. This result should be contrasted with the Little String Theory case, arising from a stack of flat coincident NS five-branes in Type II string theory, where the Hagedorn temperature behaves [16, 17] as  $\frac{1}{\sqrt{k}}$ .

In addition, from subsection 2.2.4 we know that there is no polynomial prefactor for the exponential Hagedorn density of states.

For the membrane case, the constraints given by the validity domain of the approximations are much more lax, and our results are more robust. It is not entirely clear [5] whether the corrections to the free, quadratic, results are governed by  $\left(\frac{R}{\mu N}\right)^{3/2}$  or by  $N^{1/2} \left(\frac{R}{\mu N}\right)^{3/2}$ . In any case, that magnitude might be small while  $R$  being either large or small in the  $N \rightarrow \infty$  limit. The momentum of the membrane,  $p^+$ , and its (classical) radius

$$r_2 = \frac{1}{6} \frac{\mu N}{R} \quad (5.8)$$

should be large, although the ambiguity just mentioned does not allow us to decide whether they diverge in the large  $N$  limit or whether they can be held fixed.

## 5.2 String Theory Perspective

The DLCQ of M-theory, giving matrix models, can be thought of as the limit of quantizations of M-theory on nearly light-like circles, when an auxiliary parameter  $R_s$  with a dimension of length vanishes,  $R_s \rightarrow 0$ . By boosting, this parameter can be taken as the radius of a spatial compactification of another M-theory [20, 21]. This latter compactification gives rise to a type IIA string theory on a certain background. Our aim in this subsection is to correlate the small  $R_s$  limit with the large  $N$  limit so as to obtain a valid and illuminating string theory description of our system.

The original M-theory background is a plane wave with parameter  $\mu$ , and it is compactified on an (almost) light-like circle with radius  $R$ . Performing an (almost) infinite boost, it can be regarded as an M-theory background of a plane wave with a different parameter  $\mu_s$ , compactified on a spatial circle with radius  $R_s$ .

More precisely, we start from a plane wave background

$$ds^2 = -2dx'^+ dx'^- - \left( \left( \frac{\mu_s}{3} \right)^2 (x^a)^2 + \left( \frac{\mu_s}{6} \right)^2 (x^{a'})^2 \right) (dx'^+)^2 + (dx^a)^2 + (dx^{a'})^2$$

$$F_{+123} = \mu_s \quad (5.9)$$

with an identification

$$\begin{pmatrix} t' \\ x'^{10} \end{pmatrix} \sim \begin{pmatrix} t' \\ x'^{10} + 2\pi R_s \end{pmatrix} \quad (5.10)$$

where  $x'^{\pm} = \frac{1}{2}(t' + x'^{10})$ .<sup>8</sup> This is related to our original background (1.1) by the boost

$$\begin{pmatrix} t \\ x^{10} \end{pmatrix} = \begin{pmatrix} t' \cosh \alpha - x'^{10} \sinh \alpha \\ -t' \sinh \alpha + x'^{10} \cosh \alpha \end{pmatrix} \quad (5.11)$$

or

$$x^+ = e^{-\alpha} x'^+, x^- = e^{\alpha} x'^- \quad (5.12)$$

and the plane wave parameters are related as

$$\mu = e^{\alpha} \mu_s \quad (5.13)$$

The identification in the  $(t, x^{10})$  coordinates is

$$\begin{aligned} \begin{pmatrix} t \\ x^{10} \end{pmatrix} &\sim \begin{pmatrix} t - 2\pi R_s \sinh \alpha \\ x^{10} + 2\pi R_s \cosh \alpha \end{pmatrix} \\ &= \begin{pmatrix} t \\ x^{10} \end{pmatrix} + 2\pi R \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathcal{O}(e^{-\alpha}) \end{aligned} \quad (5.14)$$

---

<sup>8</sup>This convention is different from that of the other sections by a factor of  $\sqrt{2}$ , which is not relevant for our rough estimates.

where

$$R = e^\alpha R_s \quad (5.15)$$

Thus the light-like compactification  $x^- \sim x^- + 2\pi R$  with a plane wave parameter  $\mu$  can be regarded as a limit of the spatial compactification  $x'^{10} \sim x'^{10} + 2\pi R_s$  in the infinitely boosted frame  $\alpha \rightarrow \infty$ . In this infinite boost limit,  $R_s$  is taken to be zero as  $R_s \sim e^{-\alpha}$  so that  $R$  remains finite.

From (5.13) and (5.15), and reinstituting the Planck energy scale in the original M-theory, we obtain

$$\frac{RM_p^2}{\mu} = \frac{R_s M_p^2}{\mu_s} \quad (5.16)$$

in the  $\alpha \rightarrow \infty$  limit. Instead of changing  $\mu$  to  $\mu_s$ , we prefer, following [19]-[21], to rescale the Planck length. Put otherwise, we map the original M-theory to a new M-theory, compactified on a spatial circle with radius  $R_s$ , having a plane wave parameter  $\mu$  (not  $\mu_s$ ), and a Planck energy scale  $\tilde{M}_p$ . This new theory is denoted by  $\tilde{M}$ , and all of its parameters will be written with a tilde. The advantage in doing so is that the light cone energy scale of the excitations in the original M-theory (which is  $\mu$  in our case) is kept fixed under this map. Explicitly, we demand

$$\frac{R_s \tilde{M}_p^2}{\mu} \equiv \frac{RM_p^2}{\mu} = \frac{R_s M_p^2}{\mu_s} \quad (5.17)$$

and therefore obtain that

$$\frac{M_p^2}{\mu_s} = \frac{\tilde{M}_p^2}{\mu} \quad (5.18)$$

or, from (5.13), that  $\tilde{M}_p^2 = e^\alpha M_p^2$ . To keep  $M_p$  finite and fixed, we need  $\tilde{M}_p \rightarrow \infty$ . We also rewrite (5.17) as

$$R_s \tilde{M}_p^2 = RM_p^2 \quad (5.19)$$

The  $\tilde{M}$  theory is described by  $N$  D0-branes in weakly coupled type IIA string theory, with [21] the string coupling <sup>9</sup>

$$\tilde{g}_s = R_s^{3/4} (RM_p^2)^{3/4} \rightarrow 0 \quad (5.20)$$

and string energy scale

$$\tilde{M}_s = R_s^{-1/4} (RM_p^2)^{3/4} \quad (5.21)$$

The 't Hooft coupling for the  $\tilde{M}$  theory is seen to be the same as the one in the M-theory,

$$\lambda = \left( \frac{3RM_p^2}{\mu} \right)^3 N = \left( \frac{3R_s M_p^2}{\mu_s} \right)^3 N = \left( \frac{3R_s \tilde{M}_p^2}{\mu} \right)^3 N \quad (5.22)$$

and in string units it is expressed as

$$\lambda = \tilde{g}_s N \left( \frac{3\tilde{M}_s}{\mu} \right)^3 \quad (5.23)$$

---

<sup>9</sup>These relations for flat space are valid only for a weak background. This will turn out to be the case, see below.

In the standard prescription of DLCQ in refs. [19]-[21],  $R_s$  is simply taken to be zero. We wish to consider a version of DLCQ which is appropriate for our 't Hooft limit. We scale  $R_s$  in an  $N$  dependent way in the large  $N$  limit, so that the 't Hooft coupling  $\tilde{g}_s N$  of open string theory on D0-branes is kept finite. From (5.23) this means  $\tilde{M}_s/\mu$  is kept finite, which is different from the case studied in refs. [19]-[21]. Later we will see, however, that although finite the string energy scale should be kept large compared with  $\mu$ . Then from (5.21)

$$\left(\frac{\tilde{M}_s}{\mu}\right) \sim \lambda^{1/4} (\mu N R_s)^{-1/4} \quad (5.24)$$

so we take the following large  $N$  scaling of  $R_s$  to keep  $\tilde{M}_s/\mu$  finite,

$$R_s \sim \frac{\lambda}{\mu N} \left(\frac{\mu}{\tilde{M}_s}\right)^4 \quad (5.25)$$

This is equivalent to taking the boost parameter as

$$e^{-\alpha} = \frac{R_s}{R} \sim \frac{\lambda^{2/3} M_p^2}{\mu^2} \left(\frac{\mu}{\tilde{M}_s}\right)^4 N^{2/3} \quad (5.26)$$

From (5.22) we also need to take

$$\tilde{M}_p \sim \mu \lambda^{-1/3} N^{1/3} \left(\frac{\mu}{\tilde{M}_s}\right)^{-2} \rightarrow \infty \quad (5.27)$$

The radius  $r_5$  of the spherical five-brane (5.4) is also rescaled, as a transverse direction, in the new theory,

$$\tilde{r}_5 \tilde{M}_p = r_5 M_p \quad (5.28)$$

Then

$$\tilde{r}_5^2 = \frac{M_p^2}{\tilde{M}_p^2} r_5^2 \sim \frac{R_s N}{\mu} \sim \frac{\lambda}{\mu^2} \left(\frac{\mu}{\tilde{M}_s}\right)^4 \quad (5.29)$$

Measuring the rescaled radius in the string units, we obtain

$$\tilde{r}_5^2 \tilde{M}_s^2 \sim \lambda \left(\frac{\mu}{\tilde{M}_s}\right)^2 \quad (5.30)$$

Thus the radius of the transverse spherical M five-brane, or equivalently spherical NS five-brane in IIA string theory, is kept finite and small in string units.

We need to check the validity of the flat space formulas (5.20) and (5.21) we have used. The plane wave metric in  $(t, x^{10})$  coordinate is

$$ds^2 = -\frac{1}{2}\left(1 + \frac{F^2}{4}\right)dt^2 + \frac{1}{2}\left(1 - \frac{F^2}{4}\right)(dx^{10})^2 - \frac{F^2}{4}dt dx^{10} + (dx^a)^2 + (dx^{a'})^2, \quad (5.31)$$

where

$$F^2 = \left(\frac{\mu}{3}\right)^2 (x^a)^2 + \left(\frac{\mu}{6}\right)^2 (x^{a'})^2 \quad (5.32)$$

Thus the flat space relations (5.20), (5.21) are valid when the transverse length scales we are considering are much smaller than  $1/\mu$ , thus we need

$$\mu\tilde{r}_2, \mu\tilde{r}_5 \ll 1 \quad (5.33)$$

where

$$\tilde{r}_2 \sim \frac{\mu}{R_s \tilde{M}_p^3} \quad (5.34)$$

is the size of the fuzzy membrane solution. This is the case when the 't Hooft coupling is finite and

$$\mu\tilde{r}_2 \sim \left(\frac{\mu}{\tilde{M}_s}\right)^2 \ll 1 \quad (5.35)$$

i.e.  $\mu/\tilde{M}_s$  is kept finite and small. This condition justifies the use of the D0-brane action without stringy correction.

We conclude that our 't Hooft limit is the limit where the string scale and the radius of the spherical NS five-brane are fixed. String theory in an NS five-brane background with a fixed string scale gives the little string theory (LST). The validity of our calculations is limited to small  $\lambda$  where the NS five-brane is highly curved in string units, and this prevents us from direct comparison with the results of LST on flat space [18]. However, the existence of the phase transition appears robust and is expected to occur over a range of  $\lambda$  values.

To recapitulate, the original M-theory compactification has the following parameters: the Planck energy scale  $M_p$ ,  $N$ , the null circle radius  $R$ , and the plane wave parameter  $\mu$ . We look at the compactification as the limit  $R_s \rightarrow 0$  of nearly light-like compactifications, and map, via a boost, to a string theory with the following parameters: the string energy scale  $\tilde{M}_s$ ,  $N$ , the string coupling  $\tilde{g}_s$  and  $\mu$ . Taking the  $R_s \rightarrow 0$  limit in a certain way, we are able to satisfy the assumptions underlying our approximations: in the M-theory side, those are the large  $N$  limit,  $N \rightarrow \infty$ , and the weakly coupled matrix model condition  $\lambda \ll 1$ ; we take the 't Hooft limit of fixed  $\lambda$ . In the string theory side, there are the decoupling limit condition  $\tilde{g}_s \rightarrow 0$  and the weak background condition  $\mu \ll \tilde{M}_s$  near the origin of the space, where the branes reside. We then conclude from (5.30) that the NS five-brane size is finite and small in string units,

$$\tilde{r}_5 \tilde{M}_s \ll 1 \quad (5.36)$$

### 5.3 Outlook

If we do not fix the light-cone momentum, the finite temperature partition function of the matrix model should use the rest frame energy

$$p^0 = \frac{1}{\sqrt{2}}(p^+ + p^-) \quad (5.37)$$

in the Boltzmann weight for the partition function,

$$Z = \text{Tr} e^{-\frac{\beta}{\sqrt{2}}(p^+ + p^-)} \quad (5.38)$$

The trace over  $p^+$  is a sum over  $N$  and the trace over the states of  $p^-$  is the partition function of the matrix model,

$$Z = \sum_{N=0}^{\infty} e^{-\frac{\beta N}{\sqrt{2R}}} e^{-\beta F[\beta, N]} \quad (5.39)$$

Here,  $F[\beta, N]$  is the free energy of the matrix model. The convergence of this sum for large  $N$  is determined by the free energy in the large  $N$  limit.

In the membrane states, the free energy is of order one in the large  $N$  limit and the partition sum converges. In this same limit, the  $M$  five-brane states are strongly coupled unless we take the 't Hooft limit. In that case, there is a phase transition and the behavior of the free energy changes from being of order one to order  $N^2$ . Further, when it is of order  $N^2$ , it is negative and the partition sum diverges.

This divergence could be interpreted as an instability of the finite temperature system. The logical speculation is the collapse of the system to a black hole. This is certainly what we would expect in finite temperature field theory on a Minkowski background. There, we would expect that the Jeans instability renders any theory of quantum gravity unstable at finite temperature. Indeed, in the BFSS matrix model, we do expect the free energy to grow faster than linearly in  $N$ .

## A. Fixing the Static Diagonal Gauge

The thermodynamic partition function is defined by the Euclidean path integral

$$Z = \int [dA] \dots \exp \left( - \int_0^{1/T} d\tau \frac{1}{2} \text{Tr} ((DX)^2 + \dots) \right) \quad (1.1)$$

where  $T$  is the temperature (we use units where Boltzmann's constant  $k_B = 1$ ) the variables have periodic boundary conditions,

$$A(\tau) = A(\tau + 1/T) \quad , \quad X(\tau) = X(\tau + 1/T) \quad , \quad \dots$$

This model has the gauge invariance,

$$X \rightarrow U(t) X(t) U^\dagger(t) \quad , \quad A(t) \rightarrow U(t) \left( A(t) - i \frac{d}{dt} \right) U^\dagger(t)$$

where  $U(\tau) = z U(\tau + 1/T)$  is a periodic function of time and  $z$  is constant element of the center of the group. It is present here since the gauge transform is entirely in the adjoint. In a sense, the true gauge group is the factor group  $U(N)/Z_N$ . There is a quantity, the Polyakov loop operator

$$P = \text{Tr} \left( \mathcal{P} e^{i \int_0^{1/T} d\tau A(\tau)} \right)$$

which transforms non-trivially,

$$P \rightarrow zP$$

This is interpreted as a global symmetry of the theory. Its realization is related to the gauge interaction in the theory and can be used to analyze confinement. There are two phases.

When the symmetry is good, the Polyakov loop operator averages to zero:  $\langle P \rangle = 0$ . When the symmetry is spontaneously broken, the Polyakov loop operator can have non-vanishing expectation value,  $\langle P \rangle \neq 0$ . Of course in this very low dimensional system, a discrete symmetry can only be broken in the large  $N$  limit.

The expectation value of  $P$  is interpreted as the energy of this system with one additional external fundamental representation source. Indeed, the path integral quantization of the system with such a source would replace the partition function (1.1) by  $\langle P \rangle$  and with  $n$  such sources the partition function would be replaced by  $\langle P^n \rangle$ . Thus, we interpret

$$F[T] = -T \ln (\langle P \rangle)$$

as the free energy that it would take to introduce a fundamental representation source into the system. When  $\langle P \rangle = 0$  this free energy is infinite and we say that the system is confining, whereas when  $\langle P \rangle \neq 0$  this symmetry is finite and we say that the system is deconfined.

Let us analyze the partition function (1.1) in more detail. We can fix a static gauge by enforcing the gauge condition

$$\frac{d}{d\tau} A(\tau) = 0$$

Then, taking into account the Faddeev-Popov ghost determinant, the partition function becomes

$$Z = \int [dA(\tau)] \dots \delta(dA/d\tau) \det \left[ -\frac{d}{d\tau} D_\tau \right] \exp \left( - \int_0^{1/T} d\tau \frac{1}{2} \text{Tr} ((DX)^2 + \dots) \right)$$

The measure in the path integral can be defined using the Fourier transform,

$$A(\tau) = \sqrt{T} \sum_{n=-\infty}^{\infty} A_n e^{2\pi i n T \tau}$$

The integration measure can be defined in this basis,

$$[dA(\tau)] \equiv \prod_n dA_n$$

The Faddeev-Popov determinant is a determinant over the non-zero modes,

$$\begin{aligned} \delta(dA(\tau)/d\tau) \det \left[ -\frac{d}{d\tau} D_\tau \right] &= \prod_{a,b} \prod_{n \neq 0} \delta(2\pi i n T A_n) \cdot \left[ (2\pi i n T)(2\pi i n T + i A_{ab}^{\text{adj}}) \right] \\ &= \prod_{a,b} \prod_{n \neq 0} \delta(A_n) \cdot \left[ (2\pi i n T + i A_{ab}^{\text{adj}}) \right] \end{aligned}$$

where  $A_{ab}^{\text{adj}}$  is the adjoint. Integrating the modes of  $A_n$  with  $n \neq 0$  and decomposing the measure into an integral over the eigenvalues of  $A$  and the unitary matrices necessary to diagonalize yields the measure

$$[dA] = \prod_{a=1}^N dA_a \prod_{b \neq a} (A_a - A_b)^2 [d\Omega]$$



where  $\Omega$  is the unitary matrix which diagonalizes  $A$ . Also, when  $A$  is diagonal  $A_{ab}^{\text{adj}} = A_a - A_b$ . Then, the total measure becomes

$$\begin{aligned} & \prod_{a=1}^N dA_a \prod_{b \neq a} \prod_{n=-\infty}^{\infty} (2\pi i n T + i(A_a - A_b)) \left( \prod_{n \neq 0} 2\pi i n T \right)^N \\ &= \prod_{a=1}^N \frac{dA_a}{T} \prod_{b \neq a} \prod_{n=-\infty}^{\infty} (2\pi i n T + i(A_a - A_b)) \end{aligned}$$

where we have used the zeta-function regularization to evaluate the infinite product

$$\prod_{n \neq 0} 2\pi i n T = \prod_{n=1}^{\infty} (2\pi n T)^2 = (2\pi T)^{2\zeta(0)} e^{-2\zeta'(0)} = \frac{1}{T}$$

Riemann's zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and has the values

$$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2} \ln(2\pi)$$

To take the remaining product, we use the formula

$$\begin{aligned} \prod_{n=-\infty}^{\infty} (2\pi i n T + i\phi) &= \exp \left( \sum_n \ln(2\pi i n T + i\phi) \right) \\ &= \exp \left( - \int d\phi \int d\phi \sum \frac{1}{(2\pi i n T + i\phi)^2} \right) \\ &= \exp \left( - \int d\phi \int d\phi \frac{1}{2\pi i} \oint_C dz \frac{1}{2T} \coth(z/2T) \frac{1}{(z + i\phi)^2} \right) \end{aligned}$$

where we have taken two derivatives in order to get a convergent series and the contour  $C$  encircles all of the singularities of the  $\coth z/2T$  function located at  $z = 2\pi i n T$ . Doing the contour integral by collapsing the contour onto the singularity at  $z = -i\phi$  gives

$$\begin{aligned} \prod_{n=-\infty}^{\infty} (2\pi i n T + i\phi) &= \exp \left( - \int d\phi \int d\phi \frac{d}{d(-i\phi)} \frac{1}{2T} \coth(-i\phi/2T) \right) \\ &= \exp \left( \int d\phi \left( \frac{1}{2T} \cot(\phi/2T) + \beta \right) \right) = \exp(\ln \sin(\phi/2T) + \alpha + \beta\phi) = e^{\alpha + \beta\phi} \sin(\phi/2T) \end{aligned}$$

where  $\alpha$  and  $\beta$  are arbitrary constants parameterizing the subtractions that are needed to renormalize the product. If we *arbitrarily* set both of them to zero, we get the gauge-fixed measure for integrating over gauge fields as

$$\prod_{a=1}^N \frac{dA_a}{T} \prod_{b \neq a} \left| \sin \left( \frac{A_a - A_b}{2} \right) \right|$$

The absolute value sign is there because the Faddeev-Popov determinants should always have absolute value signs.

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